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BIFURCATION OF CONCENTRATION WAVES IN FILTRATION OF BINARY SUSPENSIONS

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A one-dimensional nonstationary model of filtration of binary low-concentrated suspensions in a porous medium is considered. It is shown on the basis of the exact solution of the problem that the ratio of component partial concentration may induce two modes of concentration front displacement along the filter. In the first mode two concentration waves are formed right at the beginning, while in the second a single front is formed at the beginning which then separates at the bifurcation point into two waves. Relationships based on laws of conservation apply along the waves.

The majority of filtration systems encountered in nature and technology (colmatage of soil, clarification, etc.) are characterized by the interaction of fluid and solid phases, which results in an alteration of phase properties and, consequently in the variation of filtration conditions. As an example of such heterogeneous system we consider the filtration of low concentration suspensions in a homogeneous porous medium in which the disperse phase does not affect the macroscopic properties of the disperse medium (density, viscosity, etc.). The problem of suspensions consisting of particles of the same size and the change of their concentration owing to adhesion to the porous charge surface was considered in [1, 4]. However the most important property of filtered suspensions is the inhomogeneity of their composition, which had not been so far investigated.

Let us consider the problem of filtration of a binary suspension which forms in the pores of a stationary medium an incompressible sediment. The equation of clarification kinetics may be written in this case separately for each fraction as

$$\frac{\partial \rho_i}{\partial t} = \begin{cases} \beta_i c_i & \text{for} \quad \rho_i < \rho_{i0} \\ 0 & \text{for} \quad \rho_i = \rho_{i0} \end{cases} \qquad (i = 1, 2)$$

where c_i (x, t) is the concentration of the i-th component in the fluid phase ρ_i (x, t) is the concentration in the sediment phase, and β_i is the kinetic coefficient that defines the effectiveness of the suspended matter extraction by the porous medium.

The filtration rate based on the output is assumed constant. The assumption of the sediment incompressibility and of the van der Waals adhesion forces between particles of the suspension disperse phase allows to set $\beta_i =$ const. When the sediment concentration in a given section reaches ρ_{i0}, the local hydrodynamic pulling forces become equal to forces of molecular attraction of sediment particles, and adhesion ceases. The solution for unary suspensions in which kinetics are defined by the above equation appears in [5].

In the case of a binary mixture filtration it is necessary to take into account the reciprocal effect of fractions on the adhesion kinetics. Conditions of pre-filtration physicochemical processing of suspensions (coagulation, flocculation, sedimentation) can improve the conditions of clarification of one of the fractions, hence it is possible to set $\rho_{10} > \rho_{20}$ and $\beta_1 > \beta_2$. The equations of kinetics are of the form

$$\frac{\partial \rho_1}{\partial t} = \begin{cases} \beta_1 c_1 & \text{for } \rho_1 + \rho_2 < \rho_0 \\ 0 & \text{for } \rho_1 + \rho_2 = \rho_0 \end{cases} \quad (1)$$

$$\frac{\partial \rho_2}{\partial t} = \begin{cases} \beta_2 c_2 & \text{for } \rho_2 < \rho_{20}' \text{ и } \rho_1 + \rho_2 < \rho_0 \\ 0 & \text{for } \rho_2 = \rho_{20}' \text{ или } \rho_1 + \rho_2 = \rho_0 \end{cases} \quad (2)$$

which formalizes the following property of binary sediment formation: adhesion ceases when the limit capacity of the filter, ρ_0, which generally is not equal ρ_{10}, is reached. Adhesion of the second fraction ceases when the concentration of its sediment reaches the partial capacity $\rho_{20}' < \rho_0$.

The system is closed by the equations of component material balance

$$w \frac{\partial c_i}{\partial x} + \frac{\partial \rho_i}{\partial t} = 0 \quad (i = 1, 2) \quad (3)$$

where w is the filtration rate. (We neglect in Eq. (3) the term $\varepsilon_n \, \partial c_i/\partial t$, which for $\rho_{i0} \gg c_{i0}$ is considerably smaller than $\partial \rho_i/\partial t$, where ε_n is the filtering layer porosity). The boundary value problem for the dynamics of the front in the initially clean filter is defined by the following conditions;

$$c_i(0, t) = c_{i0}, \quad \rho_i(x, 0) = 0 \quad (4)$$

where $c_{i0} = \text{const}$ defines partial concentrations of the filter inlet cross section $x = 0$.

We introduce the dimensionless variables

$$X = \frac{\beta_2 x}{w}, \quad T = \frac{\beta_2 t}{\Gamma}, \quad \Gamma = \frac{\rho_0}{c_{10} + c_{20}}, \quad u_i = \frac{c_i}{c_{10} + c_{20}},$$

$$v_i = \frac{\rho_i}{\rho_0}, \quad \varepsilon = \frac{c_{20}}{c_{10} + c_{20}}, \quad \nu = \frac{\rho_{20}'}{\rho_0}, \quad b = \frac{\beta_1}{\beta_2}$$

where on the basis of previously made assumptions $\nu < 1$ and $b > 1$.

Integration of Eqs. (1)-(4) along the characteristic $T = 0$ yields

$$u_1(X, 0) = (1 - \varepsilon) \exp(-bX), \quad u_2(X, 0) = \varepsilon \exp(-X) \quad (5)$$

It is obvious that (5) is a solution of the system for $0 \leqslant T \leqslant T_{01}$, where T_{01} is the instant of time at which limit saturation (cut-off) for the first or second component is reached at cross section $X = 0$. Substituting (5) into (1) and (2), we obtain

$$v_1(X, T) = b(1 - \varepsilon) \, T \exp(-bX), \quad v_2(X, T) = \varepsilon T \exp(-X) \quad (6)$$

The condition of the inlet cut-off then assumes the form

$$v_2(0, T) = \nu \quad \text{or} \quad v_1(0, T) + v_2(0, T) = 1$$

Hence two variants of the cut-cff at the intake cross section $X = 0$ are possible.

1°. Cut-off due to second component

$$v_2(0, T_{01}) = \nu \quad (7)$$

2°. Cut-off due to the sum of the two components

$$v_1(0, T_{02}) + v_2(0, T_{02}) = 1 \quad (8)$$

Let us consider these variants.

1°. The substitution of (7) into (6) yields $T_{01} = \nu/\varepsilon$. At instant T_{01} the total concentration of the sediment at $X = 0$ is

$$v_1(0, T_{01}) + v_2(0, T_{01}) = \nu + \nu b \frac{1-\varepsilon}{\varepsilon}$$

Hence the criteria of occurrence of these variants are

$$\nu + \nu b \frac{1-\varepsilon}{\varepsilon} \leqslant 1 \text{ (for variants 1°)} \qquad (9)$$

$$\nu + \nu b \frac{1-\varepsilon}{\varepsilon} > 1 \text{ (for variants 2°)}$$

In dimensional quantities the criterion (9) for variant 1° may be represented in the form

$$\frac{c_{10}}{c_{20}} \leqslant \frac{\beta_2}{\beta_1}\left(\frac{\rho_0}{\rho_{20}'} - 1\right)$$

The kinetic coefficients β_1 and β_2, as well as the partial and over-all dirt content concentrations ρ_{20}' and ρ_0 respectively, are physical constants determined by the properties of suspension and porous medium. Because of this the fulfilment of condition (9) for variant 1° depends on partial concentrations c_{10} and c_{20}. When that condition is satisfied, the solution of system (1) - (4) is of the form

$$u_2 = \varepsilon \exp(-X), \quad v_2 = \varepsilon T \exp(-X) \quad \text{for} \quad 0 \leqslant T \leqslant \frac{\nu}{\varepsilon}$$

$$u_2 = \varepsilon \exp\left(-X - 1 + \frac{\varepsilon}{\nu} T\right), \quad v_2 = \nu \exp\left(-X - 1 + \frac{\varepsilon}{\nu} T\right)$$

$$\text{for} \quad \frac{\nu}{\varepsilon} \leqslant T \leqslant \frac{\nu}{\varepsilon}(X+1)$$

$$u_2 = \varepsilon, \quad v_2 = \nu \quad \text{for} \quad \frac{\nu}{\varepsilon}(X+1) \leqslant T < \infty$$

The first component moves over the partly saturated layer, hence the cut-off of each cross section with respect to the sum of components takes place after the given cross section has been saturated by the second component. The solutions for u_1 and v_1 are then of the form

$$u_1 = (1-\varepsilon)\exp(-bX), \quad v_1 = b(1-\varepsilon) T \exp(-bX)$$

$$\text{for} \quad 0 \leqslant T \leqslant \frac{1}{b}\frac{1-\nu}{1-\varepsilon}$$

$$u_1 = (1-\varepsilon)\exp[-bX - 1 + bT(1-\varepsilon)(1-\nu)^{-1}],$$

$$v_1 = (1-\nu)(1-\varepsilon)^{-1} u_1 \quad \text{for} \quad \frac{1}{b}\frac{1-\nu}{1-\varepsilon} \leqslant T \leqslant \frac{1-\nu}{1-\varepsilon}\left(X + \frac{1}{b}\right)$$

$$u_1 = 1-\varepsilon, \quad v_1 = 1-\nu \quad \text{for} \quad \frac{1-\nu}{1-\varepsilon}\left(X + \frac{1}{b}\right) \leqslant T < \infty$$

Thus in case 1° two concentration waves are formed at the intake cross section. The formation of waves needs time which for the first phase is equal $b^{-1}(1-\nu)/(1-\varepsilon)$, and for the second ν/ε. The wave of the second component moves over the layer at velocity $\sigma_2 = \varepsilon/\nu$, and simultaneously with this a wave in the sediment phase moves in-phase with the wave in the fluid phase. These waves satisfy the condition

$$\frac{u_2}{\varepsilon} = \frac{v_2}{\nu} \quad \left(T > \frac{\nu}{\varepsilon}\right) \qquad (10)$$

The first component wave moves along the filter at velocity $\sigma_1 = (1 - \varepsilon) / (1 - \nu) < \sigma_2$. In-phase with it moves the sediment wave. Condition

$$\frac{u_1}{1-\varepsilon} = \frac{v_1}{1-\nu} \quad \left(T > \frac{1}{b}\frac{1-\nu}{1-\varepsilon}\right) \tag{11}$$

is satisfied along these waves.

2°. When at cross section $X = 0$ the cut-off is effected with respect to the sum of components, then at instant T_{02} condition

$$\frac{c_{10}}{c_{20}} > \frac{\beta_2}{\beta_1}\left(\frac{\rho_0}{\rho_{20}'} - 1\right)$$

is satisfied.

The substitution of (6) into (8) yields $T_{02} = [b\,(1 - \varepsilon) + \varepsilon]^{-1}$, and then $v_2\,(0, T_{02}) = \varepsilon\,[b\,(1 - \varepsilon) + \varepsilon]^{-1} < \nu$, which conforms to condition (9) for variant 2°.

For obtaining the solution of system (1) - (3) for $T > T_{02}$ we introduce function $X = X_1^*\,(T)$ which indicates the coordinate of the cut-off cross section at instant T. The conditions at that cross section are

$$u_1\,(X_1^*,\ T) = 1 - \varepsilon, \quad u_2\,(X_1^*,\ T) = \varepsilon \tag{12}$$

$$v_1\,(X_1^*,\ T) + v_2\,(X_1^*,\ T) = 1 \tag{13}$$

We seek a solution of the form

$$u_1\ (1 - \varepsilon) f_1\,(T) \exp\,(-\,bX), \quad u_2 = \varepsilon f_2\,(T) \exp\,(-X)$$
$$(f_1\,(0) = f_2\,(0) = 1)$$

where the initial conditions for f_i are implied by formulas (5).

From condition (12) we have

$$f_1\,(T) = \exp\,[bX_1^*\,(T)], \quad f_2\,(T) = \exp\,[X_1^*\,(T)]$$

and from the equation of balance (3) we obtain

$$v_1\,(X, T) = T_{02} b\,(1 - \varepsilon) \exp\,(-\,bX) + b\,(1 - \varepsilon) \exp\,(-\,bX) \int_{T_{02}}^{T} \exp\,[bX_1^*\,(\tau)]\,d\tau \tag{14}$$

$$v_2\,(X, T) = T_{02}\varepsilon \exp\,(-\,X) + \varepsilon \exp\,(-\,X) \int_{T_{02}}^{T} \exp\,[X_1^*\,(\tau)]\,d\tau \tag{15}$$

Condition (13) at the cut-off point makes it possible to obtain the integral equation of $X_1^*\,(T)$ by rewriting it in the following equivalent form for cross section X and cut-off time $T_1^*\,(X)$

$$1 = T_{02}\,[b\,(1 - \varepsilon) \exp\,(-\,bX) + \varepsilon \exp\,(-\,X)] + b\,(1 - \varepsilon) \exp\,(-\,bX) \int_{T_{02}}^{T_1^*(X)} \exp\,[bX_1^*\,(\tau)]\,d\tau + \varepsilon \exp\,(-\,X) \int_{T_{02}}^{T_1^*(X)} \exp\,[X_1^*\,(\tau)]\,d\tau \tag{16}$$

where (T_1^* is the inverse function of X_1^*

With the use of substitution $\xi = X_1^*(\tau)$ Eq. (16) can be represented as follows.

$$1 = \frac{T_1^*(X)}{T_{02}} - b^2(1-\varepsilon)\exp(-bX)\int_0^X \exp(b\xi)\,T_1^*(\xi)\,d\xi - \quad (17)$$

$$\varepsilon\exp(-X)\int_0 \exp\xi T_1^*(\xi)\,d\xi$$

which after double differentiation yields the equation

$$\frac{d^2T_1^*}{dX^2} + bT_{02}\frac{dT_1^*}{dX} - bT_{02} = 0$$

$$\left(T_1^*(0) = T_{02},\quad \frac{dT_1^*(0)}{dX} = T_{02}^2[b^2(1-\varepsilon)+\varepsilon]\right)$$

whose solution is

$$T_1^*(X) = T_{02} + X + (bT_{02})^{-1}T_{02}^2\{[b^2(1-\varepsilon)+\varepsilon]-1\}\times [1-\exp(-XbT_{02})] \quad (18)$$

which satisfies the conditions appearing above in parantheses (the second condition is obtained from (17) after differentiation with respect to X). Having determined with the use of (18) the inverse function $X_1^*(T)$ we substitute it into (14) and (15) and can, then determine the solution for u_1 and v_1.

A singularity of the considered here case 2° is the possibility of solution bifurcation is that at some cross section X_0 at instant T_0 besides (13) the cut-of condition for the second fraction, viz. $v_2(X_0, T_0) = \nu$ or with allowance for (15)

$$\nu = T_{02}\varepsilon\exp(-X_0) + \varepsilon\exp(-X_0)\int_{T_{02}}^{T_0}\exp[X_1^*(\tau)]\,d\tau$$

Substituting (18), we obtain

$$X_0 = (bT_{02})^{-1}\ln\varkappa,\quad \varkappa = (1-\varepsilon)\,\varepsilon\,(b-1)(\varepsilon-\nu)^{-1}[b(1-\varepsilon)+\varepsilon]^{-1} \quad (19)$$

The necessary condition of bifurcation that obviously is implied by (19) is that

$$\varkappa > 1 \quad (20)$$

Because $b > 1 > 0$, from (20) we have

$$\varepsilon - \nu > 0 \quad (21)$$

and in this case condition (20) is equivalent to condition (9) for variant 2°. Since the last condition must be satisfied together with condition (21), there exists an interval of partial concentration of the second fraction in which the bifurcation mode

$$\nu < \varepsilon < \frac{\nu b}{1+\nu(b-1)} \quad (22)$$

is realized.

Parameter T_0 is calculated by substituting X_0 into solution (18)

$$T_0 = X_0 + \nu b^{-1}(b-1) + b^{-1}$$

Beyond the bifurcation point X_0 T_0 the solution of $X_1^*(T)$ splits and functions u_i and v_i assume the form

$$v_1(X,T) = T_{02} b(1-\varepsilon)\exp(-bX) + \qquad (23)$$

$$b(1-\varepsilon)\exp(-bX)\int_{T_{02}}^{T_0} \exp[bX_1^*(\tau)]\,d\tau + $$

$$b(1-\varepsilon)\exp(-bX)\int_{T_0}^{T} \exp[bX_{21}^*(\tau)]\,d\tau$$

$$v_2(X,T) = T_{02}\varepsilon\exp(-X) + \varepsilon\exp(-X)\int_{T_{02}}^{T_0} \exp[X_1^*(\tau)]\,d\tau +$$

$$\varepsilon\exp(-X)\int_{T_0}^{T} \exp[X_{22}^*(\tau)]\,d\tau$$

$$u_1(X,\ T) = (1-\varepsilon)\exp[-bX + bX_{21}^*(T)]$$

$$u_2(X,\ T) = \varepsilon\exp[-X + X_{22}^*(T)]$$

which is similar to that of formulas (14) and (15). The two functions $X_{21}^*(T)$ and $X_{22}^*(T)$ which appear in formulas (23) define the coordinate of the cut-off at instant T with respect to the first and second component, respectively.

Equations for X_{21}^* and X_{22}^* are obtained from the cut-off condition

$$v_1[X,\ T_{21}^*(X)] = 1-\nu, \quad v_2[X,\ T_{22}^*(X)] = \nu \qquad (24)$$

where T_{21}^* and T_{22}^* are inverse functions of X_{21}^* and X_{22}^*. Substituting the first two expressions of (23) into corresponding conditions (24), we obtain integral equations that are similar to (17) and whose solutions are of the form

$$T_{21}^*(x) = T_0 + \frac{1-\nu}{1-\varepsilon}(x-x_0), \quad T_{22}^*(x) = T_0 + \frac{\nu}{\varepsilon}(x+x_0)$$

Thus when condition (22) is satisfied in case 2°, the characteristic function $T_1^*(X)$ is transformed at the bifurcation point into two straight lines whose angles of inclination are σ_1 and σ_2 (see case 1°).

It is now possible to outline the pattern of concentration variation ahead and behind the critical cross sections, i.e. for $X < X_0$ and $X > X_0$.

For $X < X_0$ concentrations u_1 and v_1 are defined by the equations

$$u_1 = (1-\varepsilon)\exp[-bX + bX_1^*(T)]$$

$$u_2 = \varepsilon\exp[-X + X_1^*(T)]$$

Thus the pattern of concentration curves are similar : along the single wave condition

$$b^{-1}\ln u_1/(1-\varepsilon) = \ln u_2/\varepsilon$$

is satisfied, and at instant $T_1^*(X)$ the cut-off of the section takes place. At the same time the concentrations attain their initial values $1-\varepsilon$ and ε.

When $X > X_0$ the pattern of concentration curves at $T_{02} \leqslant T \leqslant T_0$ is the same as at the subcritical cross sections; however, at instant T_0 two waves are formed which propagate at velocities σ_1 and σ_2. The structure of these waves is determined by the equations $u_1 = (1 - \varepsilon) \exp [-bX + bX_{21}^* (T)]$ for $T_0 \leqslant T \leqslant T_{21}^* (X)$ and by $u_2 = \varepsilon \exp [-X + X_{22}^* (T)]$ for $T_0 \leqslant T \leqslant T_{22}^* (X)$.

Note that conditions (10) and (11) are satisfied along the concentration fronts. Downstream of the bifurcation point the pattern of concentration variation becomes simliar to that considered in case 1°. Time T_0 is required for the wave relationships (10) and (11) to establish themsleves.

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